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An analytic perturbative solution for the Kadomtsev equation for a heavy atom in a very strong magnetic field

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Abstract. In this paper, we use a perturbative procedure due to Bender *et al* to solve analytically the Kadomtsev equation for a heavy atom in a very strong magnetic field.

Kadomtsev [1] formulated a modified Thomas–Fermi model to describe the ground state of a heavy atom in a very strong magnetic field. This problem is of interest in connection with the emission of ions and electrons from pulsars. In this paper, we use a perturbative procedure due to Bender *et al* (BMPS) [2] to solve analytically the Kadomtsev equation,

$$\phi'' = (x\phi)^{1/2} \tag{1}$$

with the boundary conditions

$$\phi(0) = 1 \qquad \phi(\infty) = 0. \tag{2}$$

Here, primes denote differentiation with respect to the argument x.

Following the BMPS procedure, we replace the right-hand side of equation (1) by one which contains a parameter δ, i.e.

$$\phi'' = \phi\left(\frac{\phi}{x}\right)^{\delta} \tag{3}$$

so that equation (1) is recovered when $\delta = -\frac{1}{2}$ and $\delta = 0$ corresponds to the linear zero-order approximation. By identifying δ as the perturbation parameter, the potential ϕ is then expanded in a power series in δ

$$\phi = \phi_0 + \delta\phi_1 + \delta^2\phi_2 + \cdots. \tag{4}$$

This then leads to a set of linear equations for ϕ_n:

$$\begin{aligned} &\phi_0'' - \phi_0 = 0 \\ &\phi_1'' - \phi_1 = \phi_0 \ln\left(\frac{\phi_0}{x}\right) \\ &\phi_2'' - \phi_2 = \frac{1}{2}\phi_0\left(\ln\frac{\phi_0}{x}\right) + \phi_1\left[1 + \ln\left(\frac{\phi_0}{x}\right)\right] \end{aligned} \tag{5}$$

etc, with the boundary conditions

$$\phi_0(0) = 1 \qquad \phi_0(\infty) = 0$$

$$\phi_n(0) = 0 \qquad \phi_n(\infty) = 0 \qquad n \geqslant 1. \tag{6}$$

The solutions are

$$\phi_0 = \mathrm{e}^{-x}$$
$$\phi_1 = \mathrm{e}^{-x} \int_0^x \mathrm{d}\xi \int_\xi^\infty \mathrm{d}\zeta\, \mathrm{e}^{-2(\zeta-\xi)} \ln\left(\frac{\mathrm{e}^{-\zeta}}{\zeta}\right)$$

or

$$\phi_1 = \tfrac{1}{4}\,\mathrm{e}^{-x}[(1+2x)\ln x - x + x^2 + \mathrm{Ei}(1,2x)\,\mathrm{e}^{2x} + \gamma + \ln 2]$$

and

$$\phi_2 = \mathrm{e}^{-x} \int_0^x \mathrm{d}\xi \int_\xi^\infty \mathrm{d}\zeta\, \mathrm{e}^{-2(\zeta-\xi)}[\tfrac{1}{2}(\ln\zeta + \zeta)^2 + \tfrac{1}{4}(1 - \zeta - \ln\zeta)$$
$$\times\{(1+2\zeta)\ln\zeta - \zeta + \zeta^2 + \mathrm{Ei}(1,2\zeta)\,\mathrm{e}^{2\zeta} + \gamma + \ln 2\}] \tag{7}$$

etc, where

$$\mathrm{Ei}(n,x) \equiv \int_1^\infty \frac{\mathrm{e}^{-xt}}{t^n}\,\mathrm{d}t.$$

The boundary conditions

$$\phi_1(\infty) = 0 \qquad \phi_2(\infty) = 0 \tag{8}$$

etc, then lead to

$$\phi_1'(0) = \tfrac{1}{4} - \tfrac{1}{2}\ln 2 - \tfrac{1}{2}\gamma = -0.3851$$
$$\phi_2'(0) = -\tfrac{1}{8}\gamma^2 - \tfrac{1}{8}\gamma - \tfrac{1}{4}\gamma\ln 2 - \tfrac{1}{8}(\ln 2)^2 - \tfrac{1}{8}\ln 2 - \tfrac{13}{32} = -0.7668 \tag{9}$$

etc.

Thus, to $\mathrm{O}(\delta^2)$, the prediction for $\phi'(0)$ is

$$\phi'(0) = \phi_0'(0) + \delta\phi_1'(0) + \delta^2\phi_2'(0) \qquad \delta = -\tfrac{1}{2}$$

or

$$\phi'(0) = -0.9991 \tag{10}$$

which differs from the exact result (Banerjee *et al* [3])

$$\phi'(0) = -0.938 \tag{11}$$

by 6.5%!

Observe that while the first-order result

$$\phi'(0) = -0.8075 + \mathrm{O}(\delta^2)$$

is worse than the zeroth-order result

$$\phi'(0) = -1 + \mathrm{O}(\delta)$$

the second-order result (10) is better than the latter.

One may improve the convergence further by modifying the above procedure along the lines of Laurenzi [4] and others.

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